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A FUNCTIONAL INEQUALITY, WITH APPLICATIONS TO PRODUCTION THEORY

by

KING-TIM MAK

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ABSTRACT

A functional inequality is used in the formulation of a regularity condition on the scaling of production. This functional inequality is characterized and then applied to: ~~(i)~~ deduce a law of diminishing return; ~~(ii)~~ derive a bound on the growth of an open economy.

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A FUNCTIONAL INEQUALITY, WITH APPLICATIONS TO PRODUCTION THEORY

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1. INTRODUCTION

Functional equations have always been an important area in mathematics, and have found much applications in the physical sciences. Functional equations have become a useful technique in economic analysis; for example, in the study of aggregation, technical progress, structures of utility functions, price indices and scaling of production, etc. See Eichhorn [1978].

In the study of scaling of production, functional equations are used to formulate notions of homogenity, homotheticity and semi-homogenity etc; again, see Eichhorn [1978]. *This approach was extended to the formulation of ray-homotheticity* (Shephard and Färe [1977]) and general-homotheticity (Mak [1980-a]). In this paper, a different approach is taken. A simple functional inequality is proposed as a general condition on scaling, called regular scaling. This functional inequality is then characterized. Finally, the notion of regular scaling is applied to: (i) deduce a law of diminishing return, (ii) derive a bound on the speed of growth of an open economy.

2. REGULAR SCALING AND A FUNCTIONAL INEQUALITY

(2.1) Definition: A function $\phi : \mathbb{R}_+^n \to \mathbb{R}_+$ satisfies regular scaling if for every $\lambda \in \mathbb{R}_{++}$ there is a scalar $f(\lambda) \in \mathbb{R}_{++}$ such that

$$\phi(\lambda \cdot x) \leqq f(\lambda) \cdot \phi(x) \ , \ x \in \mathbb{R}_+^n \ . \tag{2.2}$$

Note that the condition of regular scaling is satisfied for homogenous and sub-homogenous functions. Furthermore, it is easy to show that the condition also holds for all "polynomials" of the form

$$\phi(x) = \sum_{k=1}^{K} \left(\prod_{i=1}^{n} a_{ki} x_i^{\alpha_{ki}} \right) , \ x \in \mathbb{R}_+^n \ , \ K < +\infty \ ,$$

if $\phi(x) > 0$ on $\mathbb{R}_{++}^n$.

To see how large is the class of functions which satisfy the functional inequality (2.2), it is convenient first to restrict the domain of ϕ to $\mathbb{R}_+$. With this specialization, the following function

$$\phi(x) = 3 + \sin x \ , \ x \in \mathbb{R}_+$$

shows that a regular scaling function may go up and down in value. Hence, it seems useful to delimit the class of functions further since we are mainly interested in production functions. The following properties (they are part of Shephard's [1974] weak axioms for production) may be imposed:

ϕ.1 $\phi(0) = 0$.

ϕ.2 $\phi(x)$ is bounded if x is bounded.

$\phi.3$ ϕ is non-decreasing (on $\mathbb{R}_+$).

$\phi.4$ ϕ is upper-semi-continuous.

In the following, a characterization of the class of $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying regular scaling is given. But, first note that

(2.3) if $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies regular scaling and $\phi.1$ but $\phi(x) = 0$ for some $x > 0$, then $\phi \equiv 0$.

When ϕ is taken as a production function, (2.3) may be somewhat too restrictive a condition. However, the properties of a production function already imply much of regular scaling in the following sense:

(2.4) Proposition: Suppose $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies $\phi.2$, $\phi.3$ and $\phi.4$. If $\phi(x)$ is positive over a compact interval $[a,b]$ where $a > 0$, then for each $\lambda \in \mathbb{R}_{++}$ there exists a scalar $f(\lambda) \in \mathbb{R}_{++}$ such that (2.2) holds for all $x \in [a,b]$.

Proof: To use contra-positive argument, suppose for a $\lambda_o \in \mathbb{R}_{++}$ there is an infinite sequence $\{x^k\} \subset [a,b]$ with $\phi\left(\lambda_o \cdot x^k\right)/\phi(x^k) \uparrow +\infty$. Since $[a,b]$ is compact, there is a subsequence $\{x^j\} \subset \{x^k\}$ with $\{x^j\} \to x^o \in [a,b]$. Clearly, $\left\{\lambda_o \cdot x^j\right\} \to \lambda_o \cdot x^o$. Since $\phi(a) > 0$ by assumption and $\phi.3$ and $\phi.4$ hold,

$$+\infty = \limsup \frac{\phi\left(\lambda_o \cdot x^j\right)}{\phi(x^j)} \leq \frac{1}{\phi(a)} \limsup \phi\left(\lambda_o \cdot x^j\right) \leq \frac{\phi\left(\lambda_o \cdot x^o\right)}{\phi(a)} .$$

This contradicts the boundedness assumption $\phi.2$. ///

Note that if $\phi \not\equiv 0$, say $\phi(z) > 0$, then by the monotonicity of ϕ , ϕ is positive on every interval $[z,y]$ where $y > z$. Hence,

every non-trivial production function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies regular scaling at least over a certain range.

(2.5) Theorem: Suppose $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies ϕ.1 and ϕ.3. Then ϕ satisfies regular scaling if and only if for some scaling factor $\lambda_o \in \mathbb{R}_{++}$, and $x_o \in \mathbb{R}_{++}$, there is a scalar $\theta \in \mathbb{R}_{++}$ such that

$$\phi\left(\lambda_o^{j+1} \cdot x_o\right) \leqq \theta \cdot \phi\left(\lambda_o^j \cdot x_o\right) \tag{2.5.1}$$

for all $j \in \{0, \pm 1, \pm 2, \ldots\}$.

Proof: Without loss of generality, one may assume $\lambda_o > 1$.

The "only if" part follows directly from the definition of regular scaling.

To show the "if" part, first observe that due to ϕ.1 and ϕ.3, if $\phi(x_o) = 0$, then $\phi \equiv 0$. This is consistent with condition (2.3). Next, note that it suffices to consider only those $x \in \mathbb{R}_{++}$ since ϕ.1 is assumed. Also, $\theta \geq 1$ because λ_o is taken to be > 1 .

Consider an arbitrary $x \in \mathbb{R}_{++}$. Let $p \in \{0, \pm 1, \pm 2, \ldots\}$ be the largest integer such that $\lambda_o^p \cdot x_o \leqq x$. Since λ_o is assumed to be larger than 1, $\lambda_o \cdot x \in \left[\lambda_o^p \cdot x_o \, , \, \lambda_o^{p+2} \cdot x_o\right]$. Thus

$$\frac{\phi(\lambda_o \cdot x)}{\phi(x)} \leqq \frac{\phi\left(\lambda_o^{p+2} \cdot x_o\right)}{\phi\left(\lambda_o^n \cdot x_o\right)} = \frac{\phi\left(\lambda_o^{p+2} \cdot x_o\right)}{\phi\left(\lambda_o^{p+1} \cdot x_o\right)} \cdot \frac{\phi\left(\lambda_o^{p+1} \cdot x_o\right)}{\phi\left(\lambda_o^p \cdot x_o\right)} \leqq \theta^2 < +\infty \; .$$

Note that the above inequality is independent of the arbitrarily chosen x . Hence, by letting $f(\lambda_o) := \theta^2$, the functional inequality (2.2) holds for λ_o .

To show (2.2) holds for all $\lambda \in \mathbb{R}_{++}$, first suppose $\lambda < \lambda_o$. It follows from ϕ.3 that

$$\sup_{x \in \mathbb{R}_{++}} \frac{\phi(\lambda \cdot x)}{\phi(x)} \leqq \sup_{x \in \mathbb{R}_{++}} \frac{\phi(\lambda_o \cdot x)}{\phi(x)} \leqq f(\lambda_o) < +\infty .$$

Hence (2.2) holds for all $\lambda < \lambda_o$. Now, suppose $\lambda > \lambda_o$. For an arbitrary $x \in \mathbb{R}_{++}$, again let p be *the largest integer with* $\lambda_o^p \cdot x_o \leqq x$. Let m be the smallest integer such that $\lambda \cdot x \leqq \lambda_o^m \cdot x_o$. It may be easily shown that for $\lambda \in \left[\lambda_o^k, \lambda_o^{k+1}\right)$ (where $k \in \{1,2, \ldots\}$), the integers p and m associated with x is related by: $m - p \leq 2 + k$. Then by ϕ.3 and the regular scaling hypothesis on x_o with scaling factor λ_o

$$\frac{\phi(\lambda \cdot x)}{\phi(x)} \leqq \frac{\phi\left(\lambda_o^m \cdot x_o\right)}{\phi\left(\lambda_o^p \cdot x_o\right)} \leqq \frac{\theta^{m-p} \cdot \phi\left(\lambda_o^p \cdot x_o\right)}{\phi\left(\lambda_o^p \cdot x_o\right)} \leqq \theta^{2+k} .$$

Since the inequality on $(m - p)$ *depends only on* λ *and not on the* arbitrarily chosen x, by letting $f(\lambda) := \theta^{2+k}$ (where k is the integer depending on λ), the functional inequality (2.2) holds. ///

Theorem (2.5) is quite remarkable in demonstrating that a single scaling factor λ_o and a single sequence of points $\left\{\ldots, \frac{1}{\lambda_o} \cdot x_o, x_o, \lambda_o x_o, \ldots\right\}$ is necessary and sufficient to test if regular scaling prevails.

With the domain of ϕ taken to be $\mathbb{R}_+^n$, property ϕ.3 (which is the assumption on input disposability when ϕ is a production function) is modified to

$\phi.3'$ $\phi(\lambda\cdot x) \geqq \phi(x)$ if $\lambda \geqq 1$, $x \in \mathbb{R}^n_+$; or

$\phi.3.S$ $x \geqq y$ implies $\phi(x) \geqq \phi(y)$.

The following propositions are concerned with function ϕ having domain $\mathbb{R}^n_+$. The first two are direct analogs of (2.4) and (2.5), hence proof will be omitted.

(2.6) <u>Proposition</u>: Suppose $\phi : \mathbb{R}^n_+ \to \mathbb{R}_+$ satisfies $\phi.2$, $\phi.3'$ and $\phi.4$. Let K be a compact subset of $\mathbb{R}^n_+$. If $\text{Inf}\{\phi(x) \mid x \in K\} > 0$, then for each $\lambda \in \mathbb{R}_{++}$ there is a scalar $f(\lambda) \in \mathbb{R}_{++}$ such that (2.2) holds for all $x \in K$.

(2.7) <u>Proposition</u>: Suppose $\phi : \mathbb{R}^n_+ \to \mathbb{R}_+$ satisfies $\phi.1$ and $\phi.3'$. Then ϕ satisfies regular scaling if and only if for some scaling factor $\lambda_o \in \mathbb{R}_{++}$, there is a scalar $\theta \in \mathbb{R}_{++}$ such that

$$\phi\left(\lambda_o^{j+1}\cdot\frac{x}{\|x\|}\right) \leqq \theta\cdot\phi\left(\lambda_o^{j}\cdot\frac{x}{\|x\|}\right)$$

for all $j \in \{0,\pm1,\pm2, \ldots\}$ and each mix $x/\|x\|$.

The above characterization of regular scaling hinges on the existence of a scalar θ for which the functional inequality holds for each mix $x/\|x\|$. The author has not been able to establish conditions under which such a scalar θ exists. However, the following proposition relating the regular scaling of one mix to another is of some interest. First, as notation, for a function $\phi : \mathbb{R}^n_+ \to \mathbb{R}_+$ and a mix $x/\|x\| \in \mathbb{R}^n_{++}$, define the function

$$\alpha \in \mathbb{R}_+ \to \phi(\alpha \mid x/\|x\|) := \phi(\alpha\cdot x/\|x\|) .$$

(2.8) <u>Proposition</u>: If ϕ.3.S holds for a function $\phi : \mathbb{R}^n_+ \to \mathbb{R}_+$ and for some mix $x/\|x\| > 0$, the function $\phi(\cdot \mid y/\|x\|) \not\equiv 0$ and satisfies regular scaling, then $\phi(\cdot \mid y/\|y\|)$ also satisfies regular scaling for every mix $y/\|y\| > 0$.

<u>Proof</u>: Suppose the mix $x/\|x\| > 0$ has $\phi(\cdot \mid x/\|x\|)$ satisfying regular scaling while the mix $y/\|y\| > 0$ has $\phi(\cdot \mid y/\|y\|)$ violating regular scaling. *For each* $\beta > 0$ *representing an element on the ray* $\{\tilde{\beta}\cdot y/\|y\| \mid \tilde{\beta} > 0\}$, define

$$\alpha^1(\beta) := \max\,\{\sigma > 0 \mid \sigma\cdot x/\|x\| \leqq \beta\cdot y/\|y\|\}\ ;$$
$$\alpha^2(\beta) := \min\,\{\sigma > 0 \mid \sigma\cdot x/\|x\| \geqq \beta\cdot y/\|y\|\}\ .$$

Since $x/\|x\| > 0$ and $y/\|y\| > 0$, $\alpha^1(\beta)$ and $\alpha^2(\beta)$ are well defined *for* $\beta > 0$. *Furthermore, it is a simple geometric fact that the ratio* $\Delta := \alpha^2(\beta)/\alpha^1(\beta)$ is independent of β.

Since the function $\phi(\cdot \mid y/\|y\|)$ violates regular scaling, there exists a $\lambda^* > 1$ with

(2.8.1) $$\sup_{\beta>0} \frac{\phi(\lambda^*\beta \mid y/\|y\|)}{\phi(\beta \mid y/\|y\|)} = +\infty\ .$$

Since $\phi(\cdot \mid x/\|x\|)$ satisfies regular scaling, for the scaling factor $\lambda^*\Delta$, there is a scalar $f(\lambda^*\Delta)$ such that

(2.8.2) $$\phi(\lambda^*\Delta\sigma \mid x/\|x\|) \leqq f(\lambda^*\Delta)\cdot\phi(\sigma \mid x/\|x\|)\ ,\ \forall\sigma > 0\ .$$

Now (2.8.1) implies the existence of a $\beta^* > 0$ with $\phi(\lambda^*\beta^* \mid y/\|y\|) > f(\lambda^*\Delta)\cdot\phi(\beta^* \mid y/\|y\|)$. But by construction and ϕ.3.S

$$\phi(\alpha^1(\beta^*)\cdot x/\|x\|) \leqq \phi(\beta^*\cdot y/\|y\|) ,$$

and

$$\phi(\lambda^*\alpha^2(\beta^*)\cdot x/\|x\|) \geqq \phi(\lambda^*\beta^*\cdot y/\|y\|) .$$

Hence letting $\sigma := \alpha^1(\lambda^*)$,

$$\frac{\phi(\lambda^*\Delta\alpha^1(\lambda^*)\cdot x/\|x\|)}{\phi(\alpha^1(\lambda^*)\cdot x/\|x\|)} = \frac{\phi(\lambda^*\alpha^2(\lambda^*)\cdot x/\|x\|)}{\phi(\alpha^1(\lambda^*)\cdot x/\|x\|)} \geqq \frac{\phi(\lambda^*\beta^*\cdot y/\|y\|)}{\phi(\beta^*\cdot y/\|y\|)} > f(\lambda^*\Delta) .$$

This contradicts (2.8.2), and the proof is completed. ///

3. APPLICATION

(A.1) A Law of Diminishing Returns

Suppose $\phi : \mathbb{R}_+^n \to \mathbb{R}_+$ is a production function. The input level sets induced by ϕ are given by

$$u \in \mathbb{R}_+ \to L(u) := \{x \in \mathbb{R}_+^n \mid \phi(x) \geqq u\} .$$

The technical efficient subsets of the technology are

$$E(u) := \{x \in L(u) \mid y \leq x \Rightarrow y \notin L(u)\} , \; u \in \mathbb{R}_+ .$$

In addition to $\phi.1$, $\phi.2$, $\phi.3'$, $\phi.4$, Shephard's [1974] weak axioms for production impose the following on the production technology:

$\phi.5$ If $\phi(\bar{x}) \geqq \bar{u} > 0$, then for every $\theta > 0$ there is a $\lambda_\theta > 0$ with $\phi(\lambda_\theta \cdot \bar{x}) \geqq \theta \cdot \bar{u}$.

E $E(u)$ is bounded for each $u \in \mathbb{R}_+$.

An input factor combination $I \subset\subset \{1, \ldots, n\}$ is said to be <u>essential</u> for production if $\phi(x) = 0$ for every input x with $x_I = 0$[1] . An input factor combination $I \subset\subset \{1, \ldots, n\}$ is <u>strong limitational</u> for output if for every bound $B \in \mathbb{R}_{++}$ on x_I , $\sup \{\phi(x) \mid \|x_I\| \leqq B\} < +\infty$.

(3.1) <u>Proposition</u>: Suppose a production function ϕ satisfies Shephard's weak axioms and regular scaling. Then an input factor combination $I \subset\subset \{1, \ldots, n\}$ is strong limitational for output if and only if it is essential.

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[1] x_I denotes the components $\{x_i, i \in I\}$ of a vector x .

The above proposition in general is not true without the assumption of regular scaling (see Shephard [1970]). For a proof of the proposition, see Mak [1980-b].

(A.2) A Limit on the Growth of an Economy

Suppose an economy has a single manufactured commodity which is used both as consumption goods (u) and as inputs to production (y). The production technology of this economy is represented by a production function ϕ. The other inputs (q) to production are exogenous to the economy and primary (i.e. essential). The stream of exogenous resources is given as $(q_0, q_1, \ldots, q_t, \ldots)$. The initial endownment of the manufactured commodity is $\bar{y}$. The evolution of the economy is characterized by a program $\{(u_t, y_t) \, ; \, t = 0,1, \ldots\}$:

$$(3.2) \qquad \begin{aligned} u_0 + y_0 &= \phi(\bar{y}, q_0) \, , \\ u_t + y_t &= \phi(y_{t-1}, q_t) \, , \; t = 1,2, \ldots \, . \end{aligned}$$

Assumption: The production function ϕ of the economy satisfies regular scaling. Moreover, there exists $K \in \mathbb{R}_{++}$ such that the scaling factor $f(\lambda)$ in the functional inequality (2.2) satisfies $f(\lambda) \leqq K\lambda$ for all $\lambda > 0$.

(3.3) Proposition(2): Suppose the production technology ϕ of an economy satisfies $\phi.2$, $\phi.4$ and the above Assumption. If for some $\alpha > 0$ the sequence of exogenous primary resources satisfies $\|q_t\| \leqq \alpha^t \|q_0\|$, then given the endownment $\bar{y}$, for every program

(2)This proposition is an analog of Radner [1967, Theorem 2.1].

$\{(u_t, y_t)\}$ which is feasible (i.e. satisfies (3.2)), the sequence $\left\{(u_t + y_t)/\alpha^t\right\}$ is bounded.

Proof: Suppose to the contrary that $(u_t + y_t)/\alpha^t$ is not bounded. Then there is a subsequence of indices S such that

(3.3.1)
$$\lim_{t \in S} \alpha^t/(u_t + y_t) = 0 \text{ ; and}$$
$$0 < \frac{u_{t-1} + y_{t-1}}{\alpha^{t-1}} \leqq \frac{u_t + y_t}{\alpha^t} \text{ , for all } t \in S \text{ .}$$

Denoting $x_t := (y_{t-1}, q_t)$, it follows that

(3.3.2)
$$\frac{\|x_t\|}{u_t + y_t} \leqq \frac{y_{t-1}}{u_t + y_t} + \frac{\|q_t\|}{u_t + y_t} \leqq \frac{1}{\alpha} + \frac{\alpha^t \|q_0\|}{u_t + y_t} \text{ .}$$

Thus, $\{x_t/(u_t + y_t)\}$ is a bounded sequence and has a subsequence (with index set SS) converging to some input x^* .

By rearranging the functional inequality (2.2), it follows from the Assumption that for each t

$$\phi\left(\frac{x_t}{u_t + y_t}\right) \geqq \frac{\phi(x_t)}{f(u_t + y_t)} \geqq \frac{\phi(x_t)}{K \cdot (u_t + y_t)} = \frac{1}{K} > 0$$

where $f(\cdot)$ denotes the appropriate scaling factor. Then, by ϕ.4 (the upper-semi-continuity of ϕ), $\phi(x^*) > 0$. Let $x^* \equiv (y^*, q^*)$. The exogenous resource component q^* of the limit point x^* has

$$\|q^*\| \leqq \limsup_{t \in SS} \frac{\|q_t\|}{u_t + y_t} \leqq \lim_{t \in SS} \frac{\alpha^t \|q_0\|}{u_t + y_t} = 0 \text{ .}$$

This contradicts the fact that exogenous resources are primary (essential). ///

Remark: Proposition (3.3) generalizes the cited theorem in Radner [1967, Theorem 2.1] in the sense that the technology of the economy is not assumed to be homogenous. Although a production function was used, the notion of regular scaling may be extended to the case of production correspondences (see Mak [1980-b]) and the result of (3.3) will remain essentially unchanged.

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